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E_T-lipschitzian and E_T-kernel aggregation operators ☆

J. Jacas [a], J. Recasens [b,*]

[a] *Sec Matemàtiques i Informàtica, ETS Arquitecura de Barcelona, Universitat Politècnica de Catalunya, Diagonal 649, 08028 Barcelona, Spain*
[b] *Secció Matemàtiques i Informàtica, ETS Arquitectura del Vallès, Universitat Politècnica de Catalunya, Pere Serra 1-15, 08190 Sant Cugat del Vallès, Spain*

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Abstract

Lipschitzian and kernel aggregation operators with respect to natural T-indistinguishability operators E_T and their powers are studied. A t-norm T is proved to be E_T-lipschitzian, and is interpreted as a fuzzy point and a fuzzy map as well. Given an archimedean t-norm T with additive generator t, the quasi-arithmetic mean generated by t is proved to be the most stable aggregation operator with respect to T.

Keywords: Aggregation operator; Indistinguishability operator; Natural indistinguishability operator; Lipschitzian condition; Kernel condition

1. Introduction

Lipschitzian conditions are fulfilled by many maps and operators in fuzzy reasoning. They give stability to the system since the similarity or distance between the outputs is bounded by the corresponding one between the inputs. The lipschitzian condition appears for example in the study of fuzzy maps [9], vague algebras [6], fuzzy modifiers and fuzzy logic in the narrow sense [16], fuzzy topology [8], extensionality [9] among others and therefore it deserves a deep study.

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* Corresponding author.
E-mail addresses: joan.jacas@upc.edu (J. Jacas), j.recasens@upc.edu (J. Recasens).

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Lipschitzian aggregation operators have been studied in [3,4,13,14] considering the usual metric on the unit interval. This paper studies the lipschitzian condition of aggregation operators in a broader sense, i.e. with respect to natural indistinguishability operators E_T and their powers E_T^p (see definitions below) so that an aggregation operator h is E_T^p-lipschitzian when for all $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n \in [0,1]\,T(E_T^p(x_1, y_1), \ldots, E_T^p(x_n, y_n)) \leqslant E_T(h(x_1, x_2, \ldots, x_n), h(y_1, y_2, \ldots, y_n))$. The meaning is that from similar values we obtain a similar aggregation. The use of E_T and E_T^p assumes the selection of a specific t-norm T and therefore the selection of a particular family of logics where the semantics of the conjunction and the bi-implication are given by T and E_T.

Let us suppose that we are studying or modeling a problem or situation in which a specific t-norm is involved. For instance, a fuzzy system with a t-norm T used to model the logical conjunction (the corresponding disjunction and implication will then probably be related to T). In this case, it seems desirable that the aggregation operators in this particular setting should be related to T in some sense. In particular, it seems more reasonable that lipschitzian conditions of the aggregation operators are defined by means of the bi-implication E_T associated to T than using the usual metric of the unit interval. In fact, as it will be seen in this paper, when T is the Lukasiewicz t-norm, the E_T-lipschitzian condition coincides with the 1-lipschitzian condition with the usual metric on $[0,1]$; not surprisingly, due to the relation between E_T and the usual metric on $[0,1]$ in this case. In other words, the definition of lipschitzian aggregation operator given in [13] tacitly assumes the use of (a logic based on) the Lukasiewicz t-norm.

Among other results, it will be proved that if T is a continuous archimedean t-norm with additive generator t and h_t the quasi-arithmetic mean generated by t $\left(h_t(x_1, x_2, \ldots, x_n) = t^{-1}\left(\frac{t(x_1)+t(x_2)+\cdots+t(x_n)}{n}\right)\right)$, then h_t is the most stable aggregation operator with respect to T (Proposition 3.18). This result not only generalizes the one in [13], but it clarifies its meaning in the sense that it corresponds to the selection of Lukasiewicz t-norm.

The easy to state, but interesting property that the E_T^p-lipschitzian condition is equivalent to the extensionality of the aggregation operator is proven (Proposition 3.10).

Also the t-norm T is not only lipschitzian with respect to E_T, but it can be seen as a fuzzy point and a fuzzy map as well (Propositions 3.20 and 3.22) and an aggregation operator h is greater than or equal to T if and only if h is E_T-lipschitzian.

If in the definition of E_T-lipschitzianity we replace the t-norm T by the minimum $(\mathrm{Min}(E_T^p(x_1, y_1), \ldots, E_T^p(x_n, y_n)) \leqslant E_T(h(x_1, x_2, \ldots, x_n), h(y_1, y_2, \ldots, y_n)))$ we obtain a generalization of the kernel aggregation operators studied in [17,13]. Again, if T is the Lukasiewicz t-norm this definition is equivalent to the one given in the above mentioned references.

After a preliminary section presenting well known definitions and results on t-norms, natural indistinguishability operators and aggregation operators, Section 3 contains the main results of the paper. A last section of concluding remarks closes the paper.

2. Preliminaries

This section contains some results on t-norms and indistinguishability operators that will be needed later on in the paper. Besides well known definitions and theorems, the power T^n of a t-norm is generalized to irrational exponents in Definition 2.4 and given explicitly for continuous archimedean t-norms in Proposition 2.7.

Though many results remain valid for arbitrary t-norms and especially for left continuous ones, for the sake of simplicity we will assume continuity for the t-norms throughout the paper.

Definition 2.1. [18,11]. A continuous t-norm is a continuous map $T : [0,1] \times [0,1] \to [0,1]$ satisfying for all $x, y, z, x', y' \in [0,1]$

1. $T(x, T(y,z)) = T(T(x,y),z)$ (associativity),
2. $T(x,y) = T(y,x)$ (commutativity),
3. If $x \leqslant x'$ and $y \leqslant y'$, then $T(x,y) \leqslant T(x',y')$ (monotonicity),
4. $T(1,x) = x$.

Since a t-norm T is associative, we can extend it to an n-ary operation in the standard way

$$T(x) = x,$$
$$T(x_1, x_2, \ldots, x_n) = T(x_1, T(x_2, \ldots, x_n)).$$

In particular, $T(\overbrace{x, x, \ldots, x}^{n \text{ times}})$ will be denoted by $x_T^{(n)}$.

If T is continuous, the nth root $x_T^{(\frac{1}{n})}$ of x with respect to T is defined by

$$x_T^{(\frac{1}{n})} = \sup\left\{z \in [0,1] | z_T^{(n)} \leqslant x\right\}$$

and for $m, n \in N$, $x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)}$.

Lemma 2.2 [11]. *If* $k, m, n \in N,\ k, n \neq 0$, *then* $x_T^{(\frac{km}{kn})} = x_T^{(\frac{m}{n})}$.

Lemma 2.3. *Let* $x_1, \ldots, x_n \in (0,1]$ *and* $n \in N$. $T\left(x_{1_T}^{(\frac{1}{n})}, \ldots, x_{n_T}^{(\frac{1}{n})}\right) \neq 0$.

Proof. Let $x_i = \mathrm{Min}(x_1, \ldots, x_n)$. Then

$$T\left(x_{1_T}^{(\frac{1}{n})}, \ldots, x_{n_T}^{(\frac{1}{n})}\right) \geqslant T\left(\overbrace{x_{i_T}^{(\frac{1}{n})}, \ldots, x_{i_T}^{(\frac{1}{n})}}^{n \text{ times}}\right) = \left(x_{i_T}^{(\frac{1}{n})}\right)_T^{(n)} = x_{i_T}^{(\frac{n}{n})} = x_i \neq 0. \qquad \square$$

Assuming continuity for the t-norm T, the powers $x_T^{(\frac{m}{n})}$ can be extended to irrational exponents in a straightforward way.

Definition 2.4. If $r \in R^+$ is a positive real number, let $\{a_n\}_{n \in N}$ be a sequence of rational numbers with $\lim_{n\to\infty} a_n = r$. For any $x \in [0,1]$, the power $x_T^{(r)}$ is

$$x_T^{(r)} = \lim_{n\to\infty} x_T^{(a_n)}.$$

Continuity assures the existence of the last limit and independence of the sequence $\{a_n\}_{n \in N}$.

Let $E(T) = \{x \in [0,1] | x_T^{(2)} = x\}$ be the set of idempotent elements of T and $\mathrm{NIL}(T) = \{x \in [0,1] | x_T^{(n)} = 0 \text{ for some } n \in N\}$ the set of nilpotent elements of T.

Definition 2.5 ([18,11]). A continuous t-norm T is archimedean if and only if $E(T) = \{0,1\}$. T is called non-strict when $\mathrm{NIL}(T) = [0,1)$. Otherwise it is called strict and $\mathrm{NIL}(T) = \{0\}$.

Theorem 2.6 (Ling [15]). *A continuous t-norm T is archimedean if and only if there exists a continuous decreasing map* $t : [0,1] \to [0,\infty]$ *with* $t(1) = 0$ *such that*

$$T(x,y) = t^{[-1]}(t(x) + t(y))$$

where $t^{[-1]}$ *stands for the pseudo-inverse of t defined by*

$$t^{[-1]}(x) = \begin{cases} 1 & \text{if } x < 0 \\ t^{-1}(x) & \text{if } x \in [0, t(0)] \\ 0 & \text{otherwise.} \end{cases}$$

T is strict if $t(0) = \infty$ *and non-strict otherwise.*

t is called an additive generator of T and two additive generators of the same t-norm differ only by a multiplicative constant.

Proposition 2.7. *Let T be an archimedean t-norm with additive generator t,* $x \in [0,1]$ *and* $r \in R^+$. *Then*

$$x_T^{(r)} = t^{[-1]}(rt(x)).$$

Proof. Due to continuity of t we need to prove it only for rational r.

If r is a natural number m, then trivially $x_T^{(m)} = t^{[-1]}(mt(x))$.
If $r = \frac{1}{n}$ with $n \in N$, then $x_T^{(\frac{1}{n})} = z$ with $z_T^{(n)} = x$ or $t^{[-1]}(nt(z)) = x$ and $x_T^{(\frac{1}{n})} = t^{[-1]}\left(\frac{t(x)}{n}\right)$.
For a rational number $\frac{m}{n}$,

$$x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)} = t^{[-1]}\left(mt\left(x_T^{(\frac{1}{n})}\right)\right) = t^{[-1]}\left(mt\left(t^{[-1]}\left(\frac{t(x)}{n}\right)\right)\right)$$
$$= t^{[-1]}\left(\frac{m}{n}t(x)\right). \qquad \square$$

Definition 2.8 [19]. The residuation $\vec{T}$ of a t-norm T is defined by

$$\vec{T}(x|y) = \sup\{\alpha \in [0,1] | T(x,\alpha) \leqslant y\}.$$

Definition 2.9. [19,2]. The natural T-indistinguishability E_T associated to a given t-norm T is the fuzzy relation on $[0,1]$ defined by

$$E_T(x,y) = T(\vec{T}(x|y), \vec{T}(y|x))$$

or equivalently

$$E_T(x,y) = \mathrm{Min}(\vec{T}(x|y), \vec{T}(y|x)).$$

Example 2.10

1. If T is a continuous archimedean t-norm with additive generator t, then $E_T(x,y) = t^{-1}(|t(x) - t(y)|)$ for all $x, y \in [0,1]$.
2. If T is the Lukasiewicz t-norm, then $E_T(x,y) = 1 - |x - y|$ for all $x, y \in [0,1]$.
3. If T is the Product t-norm, then $E_T(x,y) = \mathrm{Min}(\frac{x}{y}, \frac{y}{x})$ for all $x, y \in [0,1]$ where $\frac{z}{0} = 1$.
4. If T is the Minimum t-norm, then $E_T(x,y) = \begin{cases} \mathrm{Min}(x,y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$

E_T is indeed a special kind of T-indistinguishability operator (Definition 2.11) [2] and in a logical context, where T plays the role of the conjunction, E_T is interpreted as the bi-implication associated to T [7].

The general definition of T-indistinguishability operator is

Definition 2.11. [20,19]. Given a t-norm T, a T-indistinguishability operator E on a set X is a fuzzy relation $E : X \times X \to [0,1]$ satisfying for all $x, y, z \in X$

1. $E(x,x) = 1$ (reflexivity),
2. $E(x,y) = E(y,x)$ (symmetry),
3. $T(E(x,y), E(y,z)) \leqslant E(x,z)$ (T-transitivity).

Proposition 2.12 [19]. *Let μ be a fuzzy subset of X and T a continuous t-norm. The fuzzy relation E_μ on X defined for all $x, y \in X$ by*

$$E_\mu(x,y) = E_T(\mu(x), \mu(y))$$

is a T-indistinguishability operator on X.

Definition 2.13 [12]. Let E be a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E if and only if for all $x, y \in X$

$$T(E(x,y), \mu(y)) \leqslant \mu(x).$$

Proposition 2.14 [2]. *Let E be a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E if and only if for all $x, y \in X$*

$$E(x,y) \leqslant E_T(\mu(x), \mu(y)).$$

Finally, let us recall in this preliminary section the definition of aggregation operator.

Definition 2.15 [3]. An *aggregation operator* is a map $h : \bigcup_{n \in N} [0,1]^n \to [0,1]$ satisfying

1. $h(0, \ldots, 0) = 0$ and $h(1, \ldots, 1) = 1$,
2. $h(x) = x \ \forall x \in [0,1]$,
3. $h(x_1, \ldots, x_n) \leqslant h(y_1, \ldots, y_n)$ if $x_1 \leqslant y_1, \ldots, x_n \leqslant y_n$ (monotonicity).

The restriction of h to $[0,1]^n$ will be denoted by $h_{(n)}$ so that a global aggregation operator h can be split into the family of n-ary operators $(h_{(n)})_{n \in N}$.

3. E_T-lipschitzian and E_T-kernel aggregation operators

Lipschitzian and kernel aggregation operators with respect to natural T-indistinguishability operators E_T and their powers are a special kind of aggregation operator that generalizes the definitions of [13,17]. Their interest is in the fact that they are stable operators in the sense that the similarity between the aggregation of two n-tuples is bounded by the similarity between them.

It is interesting to point out that the lipschitzian and kernel conditions are equivalent to extensionality (Propositions 3.10 and 3.25).

Among other results, it will be proved that a t-norm T is E_T-lipschitzian and moreover the maps $T_{(n)}$ can be interpreted as fuzzy points of $[0,1]^n$ and as fuzzy maps from $[0,1]^k$ to $[0,1]^{n-k}$.

Also quasi-arithmetic means are proved to be the most stable aggregation operators in the sense stated before Proposition 3.18.

In this paper we will consider the lipschitzain condition with respect to the powers E_T^p, $p > 0$ of the natural indistinguishability operators, defined in Corollary 3.6.

Proposition 3.1 [12]. *Let E be a T-indistinguishability operator on a set X. The fuzzy relation E^n defined by*

$$E^n(x,y) = T(\overbrace{E(x,y),\ldots,E(x,y)}^{n\ \text{times}}) \quad \forall x,y \in X$$

is a T-indistinguishability operator.

Corollary 3.2 [5]. *Let E_T be the natural T-indistinguishability operator on* $[0,1]$ *associated to T. E_T^n is a T-indistinguishability operator.*

The powers E_T^n of the natural T-indistinguishability operators have been studied in relation with antonymy and fuzzy partitions in [5].

Proposition 3.3. *Let E be a T-indistinguishability operator on a set X. $E^{\frac{1}{n}}$ is a T-indistinguishability operator on X.*

Proof. Reflexivity and symmetry are trivial.

Transitivity: If $E^{\frac{1}{n}} = F$, then $F^n = E$. Since E is a T-indistinguishability operator, $\forall x,y,z \in X$

$$F^n(x,z) \geqslant T(F^n(x,y),F^n(y,z)) = (T(F(x,y),F(y,z)))_T^{(n)},$$
$$(F^n(x,z))_T^{\frac{1}{n}} \geqslant ((T(F(x,y),F(y,z)))_T^{(n)})_T^{\frac{1}{n}}$$

and from Lemma 2.2

$$F(x,z) \geqslant T(F(x,y),F(y,z)). \qquad \square$$

Corollary 3.4. *Let E_T be the natural T-indistinguishability operator on* $[0,1]$ *associated to T. $E_T^{\frac{1}{n}}$ is a T-indistinguishability operator.*

Corollary 3.5. *Let E be a T-indistinguishability operator on a set X. $E^{\frac{m}{n}}$ is a T-indistinguishability operator on X.*

Proof. Propositions 3.1 and 3.3. □

Corollary 3.6. *Let E_T be the natural T-indistinguishability operator on $[0,1]$ associated to T. $E_T^{\frac{m}{n}}$ is a T-indistinguishability operator.*

Continuity of the t-norm T allows us to extend the powers of a T-indistinguishability operator to positive irrational numbers in the same way as in Definition 2.4.

Example 3.7

1. If T is a continuous archimedean t-norm with additive generator t, then $E_T^p(x,y) = t^{[-1]}(p|t(x) - t(y)|)$ for all $x, y \in [0,1]$.
2. If T is the Lukasiewicz t-norm, then $E_T^p(x,y) = \mathrm{Max}(0, 1 - p|x - y|)$ for all $x, y \in [0,1]$.
3. If T is the Product t-norm, then $E_T^p(x,y) = (\mathrm{Min}(\frac{x}{y}, \frac{y}{x}))^p$ for all $x, y \in [0,1]$ where $\frac{z}{0} = 1$.
4. If T is the Minimum t-norm, then $E_T^p(x,y) = E_T(x,y)$ for all $x, y \in [0,1]$.

With the previous results we can relax or strengthen the equivalence relations. Indeed, $E_T^p \leqslant E_T^q$ if and only if $p \geqslant q$.

The next definition generalizes the lipschitzian condition given in [13] to the use of general indistinguishability operators E on $[0,1]$. In this paper we will only consider powers E_T^p of the natural indistinguishability operators.

Definition 3.8. Let E be a T-indistinguishability operator on $[0,1]$ and h an aggregation operator. h is E-lipschitzian if and only if $\forall n \in N$, $\forall x_1, \ldots, x_n, y_1, \ldots, y_n \in [0,1]$

$$T(E(x_1,y_1), \ldots, E(x_n,y_n)) \leqslant E_T(h(x_1, \ldots, x_n), h(y_1, \ldots, y_n)).$$

This means that if x_i and y_i are indistinguishable for $i = 1, \ldots, n$, then the aggregation of $x_1, \ldots, x_n$ should be indistinguishable or equivalent to the aggregation of $y_1, \ldots, y_n$.

Let us recall that if we have several T-indistinguishability operators $E_1, \ldots, E_n$ defined on different universes $X_1, \ldots, X_n$, there are several ways to define a T-indistinguishability operator on $X_1 \times \cdots \times X_n$.

Proposition 3.9 [12]. *Let $E_1, \ldots, E_n$ be T-indistinguishability operators on $X_1, \ldots, X_n$ respectively. Then the two fuzzy relations $T(E_1, \ldots, E_n)$ and $\mathrm{Min}(E_1, \ldots, E_n)$ on $X_1 \times \cdots \times X_n$ defined for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X_1 \times \cdots \times X_n$ by*

$$T(E_1, \ldots, E_n)((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = T(E_1(x_1,y_1), \ldots, E_n(x_n,y_n))$$

and

$$\mathrm{Min}(E_1, \ldots, E_n)((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \mathrm{Min}(E_1(x_1,y_1), \ldots, E_n(x_n,y_n))$$

are T-indistinguishability operators on $X_1 \times \cdots \times X_n$.

Thanks to this last proposition, E-lipschitzianity can be related to extensionality with respect to $T(\overbrace{E,\ldots,E}^{n\text{ times}})$.

Proposition 3.10. *Let E be a T-indistinguishability operator on $[0,1]$ and h an aggregation operator. h is E-lipschitzian if and only if $h_{(n)}$ (as a fuzzy subset of $[0,1]^n$) is extensional with respect to $T(\overbrace{E,\ldots,E}^{n\text{ times}})$ for all $n \in N$.*

Proof. Proposition 2.14. $\square$

Lemma 3.11. *Let T be a continuous t-norm. Then for all $x, y \in [0,1]x \geqslant y$*

$$T(x, \vec{T}(x|y)) = y.$$

The next proposition shows that a t-norm T is an E_T-lipschitzian aggregation operator.

Proposition 3.12. *Let T be a continuous t-norm. Then T is an E_T-lipschitzian aggregation operator.*

Proof. We must prove

$$T(E_T(x_1,y_1),\ldots,E_T(x_n,y_n)) \leqslant E_T(T(x_1,\ldots,x_n),T(y_1,\ldots,y_n))$$

or equivalently

$$T(E_T(x_1,y_1),\ldots,E_T(x_n,y_n)) \leqslant \mathrm{Min}(\vec{T}(T(x_1,\ldots,x_n)|T(y_1,\ldots,y_n)),\\ \vec{T}(T(y_1,\ldots,y_n)|T(x_1,\ldots,x_n))),$$

which means that

$$T(E_T(x_1,y_1),\ldots,E_T(x_n,y_n),x_1,\ldots,x_n) \leqslant T(y_1,\ldots,y_n) \tag{1}$$

and

$$T(E_T(x_1,y_1),\ldots,E_T(x_n,y_n),y_1,\ldots,y_n) \leqslant T(x_1,\ldots,x_n) \tag{2}$$

If $x_i \leqslant y_i$, then $E_T(x_i,y_i) = \vec{T}(y_i|x_i)$ and $T(E_T(y_i,x_i),x_i) \leqslant y_i$.

If $x_i \geqslant y_i$, then $E_T(x_i,y_i) = \vec{T}(x_i|y_i)$ and $T(E_T(x_i,y_i),x_i) = y_i$.

So inequality (1) holds.

Inequality (2) can be proved in a similar way. $\square$

Note that if $x_i \leqslant y_i$ for all $i = 1,\ldots n$, then $T(E_T(x_1,y_1),\ldots,E_T(x_n,y_n)) = E_T(T(x_1,\ldots,x_n),T(y_1,\ldots,y_n))$. Since for every t-norm different from the Minimum $E_T^p < E_T^q$ if $p > q$, we have that $T \neq \mathrm{Min}$ is not E_T^p-lipschitzian for $p < 1$.

If T is a continuous archimedean t-norm, the E_T^p-lipschitzian property translates to a classical lipschitzian condition.

Proposition 3.13. *Let T be a continuous archimedean t-norm with additive generator t, $p \in [0,1]$ and h an aggregation operator. Then h is E_T^p-lipschitzian if and only if $\forall n \in N$, $\forall x_1, \ldots, x_n, y_1, \ldots, y_n \in [0,1]$*

$$p|t(x_1) - t(y_1)| + \cdots + p|t(x_n) - t(y_n)| \geqslant |t(h(x_1, \ldots x_n)) - t(h(y_1, \ldots, y_n))|. \tag{3}$$

Proof

$$\begin{aligned}
&t^{[-1]}(t(t^{-1}(p|t(x_1) - t(y_1)|)) + \cdots + t(t^{-1}(p|t(x_n) - t(y_n)|))) \\
&\quad \leqslant t^{-1}(|t(h(x_1, \ldots, x_n)) - t(h(x_1, \ldots, x_n))|), \\
&t^{[-1]}(p|t(x_1) - t(y_1)| + \cdots + p|t(x_n) - t(y_n)|) \leqslant t^{-1}(|t(h(x_1, \ldots, x_n)) - t(h(x_1, \ldots, x_n))|), \\
&p|t(x_1) - t(y_1)| + \cdots + p|t(x_n) - t(y_n)| \geqslant |t(h(x_1, \ldots, x_n)) - t(h(x_1, \ldots, x_n))|. \qquad \square
\end{aligned}$$

The last proposition says that for all $n \in N$ the map $H : [0, t(0)]^n \to [0, t(0)]$ defined by

$$H(x_1, \ldots, x_n) = t(h(t^{-1}(x_1), \ldots, t^{-1}(x_n)))$$

is a p-lipschitzian map.

Also note that if T is the Lukasiewicz t-norm, then (3) is the definition of the Lipschitz property in [13], so that Definition 3.8 contains the one in [13] as a particular case.

If an aggregation operator h is E_T^p-lipschitzian, it may happen that for different values of n the corresponding n-ary operators $h_{(n)}$ may satisfy the lipschitzian conditions for different values of p [3, p. 12].

In the next proposition it is proved that if h is what we define as a subidempotent aggregation operator, then $h_{(n)}$ cannot be E_T^p-lipschitzian for $p < \frac{1}{n}$.

Definition 3.14. An aggregation operator is subidempotent if and only if for all $x \in [0,1]$ and $n \in N$, $h(\overbrace{x, \ldots, x}^{n \text{ times}}) \leqslant x$.

Proposition 3.15. *Let $T \neq \mathrm{Min}$ be a t-norm, h a subidempotent aggregation operator and $n \in N$. If $h_{(n)}$ is E_T^p-lipschitzian, then $p \geqslant \frac{1}{n}$.*

Proof. If $h_{(n)}$ is E_T^p-lipschitzian, then in particular, for $x \in X$

$$T((\overbrace{E_T^p(1,x), \ldots, E_T^p(1,x)}^{n \text{ times}}) \leqslant E_T(h(\overbrace{1, \ldots, 1}^{n \text{ times}}), h(\overbrace{x, \ldots, x}^{n \text{ times}}))$$

and so

$$x_T^{(pn)} \leqslant h(\overbrace{x, \ldots, x}^{n \text{ times}}) \leqslant x$$

which holds if and only if $pn \geqslant 1$ or equivalently, if and only of $p \geqslant \frac{1}{n}$. $\square$

If T is a non-strict continuous archimedean t-norm the subidempotent property can be dropped.

Proposition 3.16. *Let T be a non-strict continuous archimedean t-norm with additive generator t, h an aggregation operator and $n \in N$. If $h_{(n)}$ is E_T^p-lipschitzian, then $p \geqslant \frac{1}{n}$.*

Proof. Putting $x_i = 1$ and $y_i = 0$ for all $i = 1, \ldots, n$ in Proposition 3.13, we get

$$p|t(1) - t(0)| + \cdots + p|t(1) - t(0)| \geqslant |t(1) - t(0)|,$$
$$npt(0) \geqslant t(0)$$

or

$$p \geqslant \frac{1}{n}. \qquad \square$$

Quasi-arithmetic means are very popular aggregation operators that generalizes the arithmetic mean.

Definition 3.17. [1,11]. m is a quasi-arithmetic mean in $[0, 1]$ if and only if there exists a continuous monotonic map $t : [0, 1] \to [-\infty, \infty]$ such that for all $n \in N$ and $x_1, \ldots, x_n \in [0, 1]$

$$m(x_1, \ldots x_n) = t^{-1}\left(\frac{t(x_1) + \cdots + t(x_n)}{n}\right)$$

m is continuous if and only if Ran $t \neq [-\infty, \infty]$.

In [3], it has been proved that the arithmetic mean is the only aggregation operator h whose n-ary maps $h_{(n)}$ are $\frac{1}{n}$-lipschitzian. Proposition 3.18 generalizes this result to arbitrary quasi-arithmetic means.

Considering that the lipschitzian condition gives stability to the system in the sense that it does not allow brisk changes, we can say that for a given t-norm T an aggregation operator h is more stable than another one h' if h is E_T^p-lipschitzhian while h' is not. Next result states that quasi-arithmetic means are the most stable aggregation operators.

Proposition 3.18. *Let T be a continuous archimedean t-norm with additive generator t and m_t the quasi-arithmetic mean generated by t.*

- (a) *For every $n \in N m_{t(n)}$ is E_T^p-lipschitzian if and only if $p \geqslant \frac{1}{n}$.*
- (b) *If h is an aggregation operator such that $h_{(n)}$ is $E_T^{\frac{1}{n}}$-lipschitzian for any $n \in N$, then $h = m_t$.*

Proof

(a) $m_{t(n)}$ is E_T^p-lipschitzian if and only if

$$p|t(x_1) - t(y_1)| + \cdots + p|t(x_n) - t(y_n)|$$
$$\geqslant \left| t\left(t^{-1}\left(\frac{t(x_1) + \cdots + t(x_n)}{n}\right)\right) - t\left(t^{-1}\left(\frac{t(y_1) + \cdots + t(y_n)}{n}\right)\right)\right|$$

or equivalently

$$p|t(x_1) - t(y_1)| + \cdots + p|t(x_n) - t(y_n)| \geqslant \frac{1}{n}|t(x_1) - t(y_1) + \cdots + t(x_n) - t(y_n)|.$$

This inequality is satisfied if $p \geqslant \frac{1}{n}$. If $x_i \geqslant y_i$ for all $i = 1, \ldots, n$, then equality is attained for $p = \frac{1}{n}$.

(b) Let h be an aggregation operator with $h_{(n)}$ $E_T^{\frac{1}{n}}$-lipschitzian. Putting in particular $y_i = 1$ for all $i = 1, \dots, n$, we get

$$\frac{1}{n}|t(x_1) - t(1)| + \cdots + \frac{1}{n}|t(x_n) - t(1)| = |t(h(x_1, \dots, x_n)) - t(h(1, \dots, 1))|$$

or equivalently

$$\frac{1}{n}t(x_1) + \cdots + \frac{1}{n}t(x_n) = t(h(x_1, \dots, x_n))$$

and

$$h(x_1, \dots, x_n) = t^{-1}\left(\frac{t(x_1) + \cdots + t(x_n)}{n}\right). \qquad \square$$

In Proposition 3.12, we have proved that a t-norm T is E_T-lipschitzian. In fact, $T_{(n)}$ can also be seen as a fuzzy point of $[0,1]^n$ and a fuzzy map from $[0,1]^{n-1}$ into $[0,1]$.

Fuzzy points are the fuzzy subsets μ of a universe X that determine its granularity in the sense that if x and y are in μ, then x and y must be indistinguishable.

Fuzzy maps have been used in different places [6,7,10] because they take the granularity of the domain and image into account.

Definition 3.19 [10]. Let E be a T-indistinguishability operator on a set X and μ a fuzzy subset of X. μ is a fuzzy point of X with respect to E if and only if for all $x, y \in X$

$$T(\mu(x), \mu(y)) \leqslant E(x, y).$$

Proposition 3.20. *Let T be a continuous t-norm. $T_{(n)}$ is a fuzzy point of $[0,1]^n$ with respect to $T(\overbrace{E_T, \dots, E_T}^{n\ times})$.*

Proof. We have to prove that

$$T(T(x_1, \dots, x_n), T(y_1, \dots, y_n)) \leqslant T(E_T(x_1, y_1), \dots, E_T(x_n, y_n)),$$

which is an immediate consequence of

$$T(x_i, y_i) \leqslant E_T(x_i, y_i) \quad \text{for all } i = 1, \dots, n. \qquad \square$$

Definition 3.21 [10]. Let E, F be two T-indistinguishability operators on X and Y respectively and R a fuzzy set of $X \times Y$ (i.e.: $R : X \times Y \to [0,1]$). R is a fuzzy map from X to Y if and only if for all $x, x' \in X$, $y, y' \in Y$

- (a) $T(E(x, x'), F(y, y'), R(x, y)) \leqslant R(x', y')$
- (b) $T(R(x, y), R(x, y')) \leqslant F(y, y')$.

Proposition 3.22. *Let T be a continuous t-norm. $T_{(n)}$ is a fuzzy map from $[0,1]^{n-1}$ to $[0,1]$ endowed with the T indistinguishability operators $T(\overbrace{E_T, \dots, E_T}^{n-1\ times})$ and E_T, respectively.*

Proof. Let $(x_1,\ldots,x_{n-1}),(,x'_1,\ldots,x'_{n-1})\in[0,1]^{n-1}$ and $y,y'\in[0,1]$.
3.21 (a) is then

$$T(E_T(x_1,x'_1),\ldots,E_T(x_{n-1},y_{n-1}),E_T(y,y'),T(x_1,\ldots,x_{n-1},y))\leqslant T(x'_1,\ldots,x'_{n-1},y')$$

which is nothing but the extensionality (and therefore lipschitzianity) of $T_{(n)}$ with respect to $T(\overbrace{E_T,\ldots,E_T}^{n\ \text{times}})$.
3.21 (b)

$$T(T(x_1,\ldots,x_{n-1},y),T(x'_1,\ldots,x'_{n-1},y'))\leqslant T(y,y')\leqslant E_T(y,y'). \qquad \square$$

In fact, it can be proved in the same way that $T_{(n)}$ is a fuzzy map from $[0,1]^k$ to $[0,1]^{n-k}$ $(2\leqslant k\leqslant n-1)$ endowed with the T indistinguishability operators $T(\overbrace{E_T,\ldots,E_T}^{k\ \text{times}})$ and $T(\overbrace{E_T,\ldots,E_T}^{n-k\ \text{times}})$, respectively.

Kernel aggregation operators are a family of aggregation operators tightly related to lipschitzian ones. They were introduced in [17] (see also [13,3]). As the lipschitzian condition, the condition for being a kernel operator was related to the usual metric on the unit interval. It can be extended using natural indistinguishability operators in the same way as it has been done in this paper with the lipschitzian condition. Again, if the t-norm is the Lukasiewicz one, the original definition of [17] is recovered.

Let us recall the definition of kernel aggregation operator in [17].

Definition 3.23 [17]. An aggregation operator h is a kernel aggregation operator if and only if $\forall n\in N$, $\forall x_1,\ldots,x_n,y_1,\ldots,y_n\in[0,1]$

$$\mathrm{Max}(|x_1-y_1|,\ldots,|x_n-y_n|)\leqslant|h(x_1,\ldots,x_n)-h(y_1,\ldots,y_n)|.$$

This definition can be generalized using indistinguishability operators in a similar way as the lipschitzian condition.

Definition 3.24. Let E be a T-indistinguishability operator on $[0,1]$ and h an aggregation operator. h is an E-kernel aggregation operator if and only if $\forall n\in N$, $\forall x_1,\ldots,x_n,y_1,\ldots,y_n\in[0,1]$

$$\mathrm{Min}(E(x_1,y_1),\ldots,E(x_n,y_n))\leqslant E_T(h(x_1,\ldots,x_n),h(y_1,\ldots,y_n)).$$

Proposition 3.25. *Let E be a T-indistinguishability operator on $[0,1]$ and h an aggregation operator. h is an E-kernel aggregation operator if and only if $h_{(n)}$ (as a fuzzy subset of $[0,1]^n$) is extensional with respect to* $\mathrm{Min}(\overbrace{E,\ldots,E}^{n\ \text{times}})$ *for all $n\in N$.*

Proof. Proposition 2.14. $\square$

For archimedean t-norms, the kernel property can be written as follows.

Proposition 3.26. *Let T be a continuous archimedean t-norm with additive generator t, $p \in [0,1]$ and h an aggregation operator. h is E_T^p-kernel aggregation operator if and only if* $\forall n \in N$, $\forall x_1, \ldots, x_n, y_1, , , , y_n \in [0,1]$

$$\mathrm{Max}(p|t(x_1) - t(y_1)|, \ldots, p|t(x_n) - t(y_n)|) \geqslant |t(h(x_1, \ldots, x_n)) - t(h(y_1, \ldots, y_n))|. \quad (4)$$

Proof

$$\begin{aligned}&\mathrm{Min}(t^{-1}(p|t(x_1) - t(y_1)|), \ldots, t^{-1}(p|t(x_n) - t(y_n)|))\\ &\quad \leqslant t^{-1}(|t(h(x_1, \ldots, x_n)) - t(h(x_1, \ldots, x_n))|),\\ &t^{-1}(\mathrm{Max}(p|t(x_1) - t(y_1)|, \ldots, p|t(x_n) - t(y_n)|)) \leqslant t^{-1}(|t(h(x_1, \ldots, x_n)) - t(h(x_1, \ldots, x_n))|),\\ &\mathrm{Max}(p|t(x_1) - t(y_1)|, \ldots, p|t(x_n) - t(y_n)|) \geqslant |t(h(x_1, \ldots, x_n)) - t(h(x_1, \ldots, x_n))|. \quad \square\end{aligned}$$

If T is the Lukasiewicz t-norm and $p = 1$, then (4) is the definition of the kernel aggregation operator in [17].

4. Concluding remarks

In this paper lipschitzian and kernel aggregation operators with respect to natural T-indistinguishability operators E_T and their powers have been studied.

It has been proved that a t-norm T is E_T-lipschitzian, and a fuzzy point and a fuzzy map as well.

Quasi-arithmetic means m_t play an important role since they are the most stable aggregation operator with respect to T, meaning that the corresponding n-ary operators $m_{t(n)}$ are $E_T^{\frac{1}{n}}$-lipschitzian maps.

Lipschitzian and kernel properties are not only interesting for aggregation operators, but in almost any part of fuzzy reasoning and they deserve a deep study.

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